# UPPER BOUNDS FOR THE EIGENVALUES OF HESSIAN EQUATIONS

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ABSTRACT. In this paper we prove some upper bounds for the Dirichlet eigenvalues of a class of fully nonlinear elliptic equations, namely the Hessian equations.

## 1. INTRODUCTION

In this paper we deal with the eigenvalue problem of the $k$-Hessian operator, namely

$$(1.1) \qquad \begin{cases} S_k(D^2u) = \lambda(-u)^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 \leq k \leq n$, and $\Omega$ is a bounded, strictly convex, open set of $\mathbb{R}^n$, $n \geq 2$, with $C^2$ boundary. Here $S_k(D^2u)$ is the $k$-th elementary symmetric function of the eigenvalues of the Hessian matrix of $u \in C^2(\Omega)$ (see Section 2 for the precise definitions). Notice that for $k = 1$, $S_1(D^2u)$ reduces to the Laplacian operator $\Delta u$, while for $k = n$, $S_n(D^2u)$ is the Monge-Ampère operator $\det D^2u$.

Our aim is to generalize some well-known estimates involving the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet-Laplacian in $\Omega$. In this case, the Faber-Krahn inequality states that $\lambda_1(\Omega)$ attains its minimum at the ball $\Omega^\#$ with the same Lebesgue measure of $\Omega$, that is

$$(1.2) \qquad \lambda_1(\Omega^\#) \leq \lambda_1(\Omega).$$

Hence, a natural question which arises from (1.2) is to give an upper bound of $\lambda_1(\Omega)$. For example, in [13] it is proved that for a convex plane domain of area $|\Omega|$ and perimeter $P(\Omega)$,

$$(1.3) \qquad \lambda_1(\Omega) \leq 3\frac{P^2(\Omega)}{|\Omega|^2}.$$

The constant $c = 3$ is not sharp, and Pólya in [15] has shown that it can be replaced by $\pi^2/4$. Moreover, the inequality holds for any simply connected bounded open set $\Omega$ of $\mathbb{R}^2$.

Another classical result, due to Payne and Weinberger (see [14]), allows to obtain an upper bound of $\lambda_1(\Omega)$ in terms of $\lambda_1(\Omega^\#)$ and the isoperimetric deficit. More precisely, if $\Omega$ is a simply connected, bounded open set of $\mathbb{R}^2$ with smooth boundary, then

$$(1.4) \qquad \lambda_1(\Omega) \leq \lambda_1(\Omega^\#)\left[1 + C\left(\frac{P^2(\Omega)}{4\pi|\Omega|} - 1\right)\right],$$

where $C$ is a universal sharp constant, which can be explicitly determined. Hence, together with the Faber-Krahn inequality it is possible to obtain a stability estimate for $\lambda_1(\Omega)$, that is

$$(1.5) \qquad 0 \leq \frac{\lambda_1(\Omega) - \lambda_1(\Omega^\#)}{\lambda_1(\Omega^\#)} \leq C\left(\frac{P^2(\Omega)}{4\pi|\Omega|} - 1\right).$$

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Recently, an estimate of this kind, which involves an isoperimetric deficit of $\Omega$, has been obtained in the paper [2] for a larger class of operators in any dimension. In particular, the authors prove that, if $\Omega$ is a bounded convex open set of $\mathbb{R}^n$, then

$$(1.6) \qquad \frac{\lambda_{1,p}(\Omega) - \lambda_{1,p}(\Omega^\star)}{\lambda_{1,p}(\Omega)} \leq C(n, p, \Omega) \left( 1 - \frac{n^{\frac{n}{n-1}} \omega_n^{\frac{1}{n-1}} |\Omega|}{P(\Omega)^{\frac{n}{n-1}}} \right),$$

where $\lambda_{1,p}(\Omega)$ is the first Dirichlet eigenvalue for the $p$-Laplace operator, $\Omega^\star$ is the ball centered at the origin with the same perimeter of $\Omega$. As matter of fact, being $\lambda_{1,p}(\Omega^\star) \leq \lambda_{1,p}(\Omega^\#)$, together with the Faber-Krahn inequality of the $p$-Laplacian, we have that the left-hand side of (1.6) is nonnegative.

The main idea in order to prove the quoted estimates is to make use of a particular class of test functions, depending on the distance to the boundary, introduced in [13], [15] and nowadays known as web functions (see for example [5]).

The aim of the paper is to prove estimates for the eigenvalue of (1.1) in the same spirit of (1.3) and (1.6), when $\Omega$ is a bounded, strictly convex, open set with $C^2$ boundary. In particular, we show that if $1 \leq k \leq n$, a Makai-type estimate holds, namely

$$(1.7) \qquad \lambda_k(\Omega) \leq \frac{n(k+2)}{n-k+1} \frac{P(\Omega)^{k+1}}{|\Omega|^{k+2}} W_{k-1}(\Omega).$$

Here $\lambda_k(\Omega)$ denotes the eigenvalue of $S_k$ in $\Omega$, and $W_{k-1}(\Omega)$ is the $(k-1)$-th quermassintegral of $\Omega$ (see Section 2 for the precise references and definitions). In the Laplacian case, with $k = 1$ and $n = 2$, we recover exactly (1.3). In the Monge-Ampère case, it is worth to compare (1.7) with the upper bound obtained in [1] (see Remark 4.4 and Example 4.1)

Regarding to the stability estimates, our results read as follows. If $k = n$, we will prove that

$$(1.8) \qquad \frac{\lambda_n(\Omega) - \lambda_n(\Omega^*_{n-1})}{\lambda_n(\Omega)} \leq C_\Omega \left( |\Omega^*_{n-1}| - |\Omega| \right).$$

Here $\Omega^*_i, i = 0, \dots n-1$ denotes the ball centered at the origin with the same $i$-th quermassintegral of $\Omega$. Hence, in conjunction with the Faber-Krahn inequality for the Monge-Ampère operator (see [3] and [6]), the left-hand side of (1.8) is nonnegative and we have a stability estimate of $\lambda_n(\Omega)$.

In the case $1 \leq k \leq n - 1$, we will obtain that

$$(1.9) \qquad \frac{\lambda_k(\Omega) - \lambda_k(\Omega^*_k)}{\lambda_k(\Omega)} \leq C_\Omega \left( |\Omega^*_k| - |\Omega| \right).$$

Again, under suitable assumptions on $\Omega$, the above inequality, in conjunction with the Faber-Krahn inequality for $S_k$ (see [6] and Section 2.2), gives a quantitative estimate.

The paper is organized as follows. In Section 2, we recall some basic definitions of convex analysis and the main properties of the eigenvalues of $S_k$. Then, in Section 3 we prove some preliminary results necessary to prove the main results. In particular, we cannot apply directly the method of web functions, since they are not sufficiently regular in order to be used as test functions in (1.1). Then, we construct a suitable smooth approximating sequence of the distance function. Finally, in Section 4 we state precisely the main results and give the proofs.

## 2. Notation and preliminaries

Throughout the paper, we will denote with $\Omega$ a set of $\mathbb{R}^n$, $n \geq 2$ such that

(2.1)          $\Omega$ is a bounded, strictly convex, open set with $C^2$ boundary.

Given a function $u \in C^2(\Omega)$, we denote by $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the vector of the eigenvalues of $D^2u$. The $k$-Hessian operator $S_k(D^2u)$, with $k = 1, 2, \ldots, n$, is

(2.2) $$S_k(D^2u) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdot \lambda_{i_2} \cdots \lambda_{i_k}.$$

Hence $S_k(D^2u)$ is the sum of all $k \times k$ principal minors of the matrix $D^2u$.

The $k$-Hessian operator can be written also in divergence form, that is

(2.3) $$S_k(D^2u) = \frac{1}{k} \sum_{i,j=1}^{n} (S_k^{ij} u_i)_j,$$

where $S_k^{ij} = \frac{\partial S_k(D^2u)}{\partial u_{ij}}$ (see for instance [20], [21], [22]).

Well known examples ok $k$-Hessian operators are $S_1(D^2u) = \Delta u$, the Laplace operator, and $S_n(D^2u) = \det(D^2u)$, the Monge-Ampère operator.

It is well-known that $S_1(D^2u)$ is elliptic. This property is not true in general for $k > 1$. As matter of fact, the $k$-Hessian operator is elliptic when it acts on the class of the so-called $k$-convex function, defined below.

**Definition 2.1.** Let $\Omega$ as in (2.1). A function $u \in C^2(\Omega)$ is a $k$-convex function (strictly $k$-convex) in $\Omega$ if

(2.4)          $S_j(D^2u) \geq 0 \, (> 0) \quad \text{for } j = 1, \ldots, k.$

We denote the class of $k$-convex functions in $\Omega$ such that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$ by $\Phi_k^2(\Omega)$.

If we define with $\Gamma_k$ the following convex open cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, S_2(\lambda) > 0, \ldots, S_k(\lambda) > 0\},$$

in [9] it is proven that $\Gamma_k$ is the cone of ellipticity of $S_k$. Hence the $k$-Hessian operator is elliptic with respect to the $k-$convex functions.

If $u$ is $k$-convex, the following Newton inequalities hold:

(2.5) $$\frac{\Delta u}{n} \geq \ldots \geq \left(\binom{n}{k-1}^{-1} S_{k-1}(D^2u)\right)^{\frac{1}{k-1}} \geq \left(\binom{n}{k}^{-1} S_k(D^2u)\right)^{\frac{1}{k}}.$$

By (2.5) it follows that the $k$-convex functions equal to zero on the boundary of $\Omega$ are negative in $\Omega$.

We go on by recalling some definitions of convex analysis which will be largely used in next sections. Standard references for this topic are [4], [18].

### 2.1. Quermassintegrals and Alexsandrov-Fenchel inequalities.
Let $K$ be a convex body, and let $\rho > 0$. We denote with $|K|$ the Lebesgue measure of $K$, with $P(K)$ the perimeter of $K$ and with $\omega_n$ the measure of the unit ball in $\mathbb{R}^n$.

The well-known Steiner formula for the Minkowski sum is

$$|K + \rho B_1| = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i.$$

The coefficient $W_i(K)$, $i = 0, \ldots, n$ is known as the $i$-th quermassintegral of $K$. Some special cases are $W_0(K) = |K|$, $nW_1(K) = P(K)$, $W_n(K) = \omega_n$. If $K$ as $C^2$ boundary, with nonvanishing Gaussian curvature, the quermassintegrals can be related to the principal curvatures of $\partial K$. Indeed, in such a case

$$W_i(K) = \frac{1}{n} \int_{\partial K} H_{i-1} d\mathcal{H}^{n-1}, \quad i = 1, \ldots n.$$

Here $H_j$ denotes the $j$-th normalized elementary symmetric function of the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of $\partial K$, that is $H_0 = 1$ and

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n-1} \kappa_{i_1} \cdots \kappa_{i_j}, \quad j = 1, \ldots, n-1.$$

An immediate computation shows that if $B_R$ is a ball of radius $R$, then

(2.6) $$W_i(B_R) = \omega_n R^{n-i}, \quad i = 0, \ldots, n.$$

A Steiner formula holds true also for every quermassintegral, that is

$$W_p(K + \rho B_1) = \sum_{i=0}^{n-p} \binom{n-p}{i} W_{p+i}(K) \rho^i, \quad p = 0, \ldots, n-1.$$

This formula immediately gives that

(2.7) $$\lim_{\rho \to 0^+} \frac{W_p(K + B_\rho) - W_p(K)}{\rho} = (n-p)W_{p+1}(K), \quad p = 0, \ldots, n-1.$$

The Aleksandrov-Fenchel inequalities state that

(2.8) $$\left( \frac{W_j(K)}{\omega_n} \right)^{\frac{1}{n-j}} \geq \left( \frac{W_i(K)}{\omega_n} \right)^{\frac{1}{n-i}}, \quad 0 \leq i < j \leq n-1,$$

where the inequality is replaced by an equality if and only if $K$ is a ball.

In what follows, we use the Aleksandrov-Fenchel inequalities for particular values of $i$ and $j$. If $i = 1$, and $j = k - 1$, we have that

(2.9) $$W_{k-1}(K) \geq \omega_n^{\frac{k}{n-1}} n^{\frac{n-k+1}{n-1}} P(K)^{\frac{n-k+1}{n-1}}, \quad 3 \leq k \leq n-1.$$

When $i = 0$ and $j = 1$, we have the classical isoperimetric inequality:

$$P(K) \geq n\omega_n^{\frac{1}{n}} |K|^{1-\frac{1}{n}}.$$

Moreover, if $i = k - 1$, and $j = k$, we have

$$W_k(K) \geq \omega_n^{\frac{1}{n-k+1}} W_{k-1}(K)^{\frac{n-k}{n-k+1}}.$$

2.2. **Eigenvalue problems for** $S_k$**.** Let us consider the eigenvalue problem associated to $k$-Hessian operator, namely

(2.10) $$\begin{cases} S_k(D^2 u) = \lambda(-u)^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The following existence result holds (see [11] for $k = n$, and [22], [7] in the general case):

**Theorem 2.1.** *Let $\Omega$ as in (2.1). Then, there exists a positive constant $\lambda_k(\Omega)$ depending only on $n, k$, and $\Omega$, such that problem (2.10) admits a solution $u \in C^2(\Omega) \cap C^{1,1}(\overline{\Omega})$, negative in $\Omega$, for $\lambda = \lambda_k(\Omega)$ and $u$ is unique up to positive scalar multiplication. Moreover, $\lambda_k(\Omega)$ has the following variational characterization:*

$$(2.11) \qquad \lambda_k(\Omega) = \min_{\substack{u \in \Phi_k^2(\Omega) \\ u \neq 0}} \frac{\int_\Omega (-u) S_k(D^2 u)\, dx}{\int_\Omega (-u)^{k+1}\, dx}.$$

As matter of fact, if $k < n$ the above theorem holds under a more general assumption on $\Omega$, namely requiring that $\Omega$ is strictly $k$-convex (see [22], [7]).

We refer to $\lambda_k(\Omega)$ and $u$, respectively, as the eigenvalue and eigenfunction of $k$-Hessian operator. Moreover, given a function $u \in \Phi_k^2(\Omega)$, the quantity $\int_\Omega (-u) S_k(D^2 u)\, dx$ is known as $k$-Hessian integral. Using the divergence form of $S_k$ and the coarea formula, in [19] it is proved that

$$(2.12) \qquad \int_\Omega (-u) S_k(D^2 u)\, dx = \int_0^{\|u\|_\infty} dt \int_{\{u = -t\}} H_{k-1}(\{u = -t\}) |Du|^k d\mathcal{H}^{n-1}.$$

Hence, the variational formulation (2.11) can be written in terms of (2.12).

As matter of fact, we observe that if $k = 1$, or $k = n$, $\lambda_k(\Omega)$ coincides respectively with the first eigenvalue of the Laplacian operator, or with the eigenvalue of Monge-Ampère operator.

If $k = 1$, the well-known Faber-Krahn inequality states that

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\#),$$

where $\Omega^\#$ is the ball centered at the origin with the same Lebesgue measure of $\Omega$. Moreover, the equality holds if $\Omega = \Omega^\#$.

In [3], [6] it is proved that if $k = n$ and $\Omega$ is a bounded strictly convex open set, then

$$(2.13) \qquad \lambda_n(\Omega) \geq \lambda_n(\Omega_{n-1}^*),$$

where $\Omega_{n-1}^*$ is the ball centered at the origin such that $W_{n-1}(\Omega) = W_{n-1}(\Omega_{n-1}^*)$. We explicitly observe that if $n = 2$, $\Omega_1^*$ is the ball with the same perimeter of $\Omega$. In general, in [6] it is proven that if $\Omega$ is a strictly convex set such that the eigenfunctions have convex level sets, then, for $2 \leq k \leq n - 1$,

$$(2.14) \qquad \lambda_k(\Omega) \geq \lambda_k(\Omega_{k-1}^*),$$

where $\Omega_{k-1}^*$ is the ball centered at the origin such that $W_{k-1}(\Omega) = W_{k-1}(\Omega_{k-1}^*)$.

The additional hypothesis on $\Omega$ seems to be natural. Indeed, for $k = 1$ this is due to the Korevaar concavity maximum principle (see [10]), while it is trivial for $k = n$. For the $k$-Hessian operators, at least in the case $n = 3$ and $k = 2$, it in [12] and [16] is proved that if $\Omega$ is sufficiently smooth, the eigenfunctions of $S_2$ have convex level sets. Up to our knowledge, the general case is an open problem.

We observe that a consequence of the Aleksandrov-Fenchel inequalities is that, for a set $\Omega$ as in (2.1), then

$$(2.15) \qquad \Omega^\# = \Omega_0^* \subseteq \Omega_1^* \subseteq \ldots \Omega_{k-1}^* \subseteq \Omega_k^* \subseteq \ldots \subseteq \Omega_{n-1}^*, \quad k = 1, \ldots n - 1,$$

with the equal sign holding if and only if $\Omega$ is a ball. Indeed, denoted by $R_k$ the radius of $\Omega_k^*$, then by (2.6) and (2.8) we have

$$R_{k-1} = \left(\frac{W_{k-1}(\Omega)}{\omega_n}\right)^{\frac{1}{n-k+1}} \le \left(\frac{W_k(\Omega)}{\omega_n}\right)^{\frac{1}{n-k}} \le R_k.$$

From (2.15) and the monotonicity of $\lambda_k(\cdot)$ with respect to the inclusion of sets, it follows that, for a set $\Omega$ such that (2.14) holds, we have

$$\lambda_k(\Omega) \ge \lambda_k(\Omega_{k-1}^*) \ge \lambda_k(\Omega_k^*) \ge \ldots \ge \lambda_k(\Omega_{n-1}^*).$$

## 3. Some useful preliminary results

Let $\Omega$ as in (2.1), and $d(x)$ the distance of a point $x \in \Omega$ to the boundary $\partial\Omega$. We denote by

$$\Omega_t = \{x \in \Omega \colon d(x) > t\}, \quad t \in [0, r_\Omega],$$

where $r_\Omega$ is the inradius of $\Omega$. The Brunn-Minkowski inequality for quermassintegrals ([18, p.339]) and the concavity of the distance function give that the function $W_k(\Omega_t)^{\frac{1}{n-k}}$ is concave in $[0, r_\Omega]$. Hence, $W_k(\Omega_t)$, $t \in [0, r_\Omega]$ is a decreasing, absolutely continuous function.

**Lemma 3.1.** *For any $0 \le p \le n-1$, and for almost every $t \in\,]0, r_\Omega[$,*

$$(3.1) \qquad -\frac{d}{dt} W_p(\Omega_t) \ge (n-p) W_{p+1}(\Omega_t),$$

*where the equality sign holds if $\Omega$ is a ball.*

*Proof.* It is not difficult to prove that, if $B_1$ is the unit ball centered at the origin, we have

$$\Omega_t + \rho B_1 \subset \Omega_{t-\rho}, \quad 0 < \rho < t,$$

and the equality holds when $\Omega$ is a ball. Since the quermassintegral $W_p(K)$ is monotone with respect to the inclusion of convex sets, the above relation and (2.7) give that

$$-\frac{d}{dt} W_p(\Omega_t) = \lim_{\rho \to 0^+} \frac{W_p(\Omega_{t-\rho}) - W_p(\Omega_t)}{\rho} \ge \lim_{\rho \to 0^+} \frac{W_p(\Omega_t + \rho B_1) - W_p(\Omega_t)}{\rho} = (n-p) W_{p+1}(\Omega_t)$$

for almost every $t \in\,]0, r_\Omega[$. $\qquad\square$

**Remark 3.1.** As matter of fact, it is well-known that the inequality (3.1) holds as an equality when $p = 0$. In such a case $W_0(\Omega_t) = |\Omega_t|$, $W_1(\Omega_t) = nP(\Omega_t)$. Moreover, using the coarea formula, and being $d \in W^{1,\infty}(\Omega)$ with $|Dd| = 1$ a.e., we have for a.e. $t \in\,]0, r_\Omega[$

$$(3.2) \qquad -\frac{d}{dt}|\Omega_t| = \int_{\{d=t\}} \frac{1}{|Dd|} d\mathcal{H}^{n-1} = P(\Omega_t).$$

An immediate consequence of Lemma 3.1 is the following result.

**Lemma 3.2.** *Let $u(x) = f(d(x))$, where $f\colon [0, +\infty[\,\to\,[0, +\infty[$ is a strictly $C^1$ function with $f(0) = 0$. Set*

$$E_t = \{x \in \Omega \colon u(x) > t\} = \Omega_{f^{-1}(t)}.$$

*Then, for $0 \le p \le n-1$, and for a.e. $t \in\,]0, r_\Omega[$,*

$$-\frac{d}{dt} W_p(E_t) \ge (n-p)\frac{W_{p+1}(E_t)}{|Du|_{u=t}}.$$

We conclude the Section with other two results which will be used in next sections. The first one concerns an integral inequality, while the second gives an approximation of the distance with suitable smooth functions.

**Lemma 3.3.** *Let $f \colon [0, +\infty[ \to \mathbb{R}$ a $C^1$ nondecreasing function. Denoted with $P(t) = P(\Omega_t)$ and with $r_\Omega$ the inradius of $\Omega$, then*

$$(3.3) \qquad \int_0^{r_\Omega} f(t)P(t)\,dt \geq P(\Omega) \int_0^{|\Omega|/P(\Omega)} f(t)\,dt.$$

*Proof.* We first observe that from (3.2) it holds that

$$|\Omega| = \int_0^{r_\Omega} P(t)dt \leq P(\Omega)r_\Omega.$$

Then we can define the auxiliary function

$$\tilde{P}(t) = \begin{cases} P(\Omega) & \text{if } 0 \leq t \leq \frac{|\Omega|}{P(\Omega)}, \\ 0 & \text{if } \frac{|\Omega|}{P(\Omega)} < t \leq r_\Omega. \end{cases}$$

It is easy to see that

$$(3.4) \qquad \int_0^s P(t)dt \leq \int_0^s \tilde{P}(t)dt, \quad \forall s \in [0, r_\Omega[,$$

and

$$(3.5) \qquad \int_0^{r_\Omega} P(t)dt = \int_0^{r_\Omega} \tilde{P}(t)dt.$$

Indeed, (3.4) is obvious if $s < |\Omega|/P(\Omega)$. Otherwise,

$$\int_0^s P(t)\,dt = |\Omega| - |\Omega_s| \leq \int_0^{|\Omega|/P(\Omega)} P(\Omega)\,dt = \int_0^s \tilde{P}(t)\,dt,$$

and we also have (3.5), since $|\Omega_{r_\Omega}| = 0$. Then, being $f$ increasing, an integration by parts shows that (3.4) and (3.5) imply (3.3). □

**Proposition 3.1.** *Suppose that $\Omega$ verifies (2.1). Then, there exists a sequence of functions $\{d_\varepsilon(x)\}_{\varepsilon > 0}$, $x \in \bar{\Omega}$, such that:*

(1) $d_\varepsilon$ concave in $\Omega$, $d_\varepsilon \equiv 0$ on $\partial\Omega$ and $d_\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega})$;
(2) $0 \leq d_\varepsilon \leq d$, and $d_\varepsilon \to d$ uniformly in $\bar{\Omega}$;
(3) $|Dd_\varepsilon| \leq 1$ in $\Omega$.

*Proof.* As well-known, the function $d$ is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} |Dw|^2 = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the standard vanishing viscosity argument, the required sequence can be obtained by solving the problems

$$(3.6) \qquad \begin{cases} \varepsilon\Delta w - |Dw|^2 + 1 = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness of a solution in $C^2(\Omega) \cap C(\bar{\Omega})$ of (3.6) can be proved by making the change of variable

$$z = \exp\left\{-\frac{w}{\varepsilon}\right\} - 1.$$

Then, $w$ is a solution to (3.6) if and only if $z \in C^2(\Omega) \cap C(\bar{\Omega})$ verifies

$$(3.7) \qquad \begin{cases} \varepsilon^2\Delta z - z = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well-known that problem (3.7) admits a unique solution $C^2(\Omega) \cap C(\bar{\Omega})$. Hence, the function $w = -\varepsilon \log(z + 1)$ is the unique solution of (3.6). For any $\varepsilon > 0$, we choose $d_\varepsilon = w$.

By comparison arguments, it is possible to show that $d_\varepsilon$ satisfies (2) and (3) (see for instance [17] for the details). Finally, the concavity of $d_\varepsilon$ follows applying the Korevaar concavity maximum principle to (3.6) (see [10]). □

## 4. Main results

In this section we state and prove the main results on upper bounds for the eigenvalue of $S_k$. For ease of reading, we organize the Section in three different subsections.

The first aim is to prove an upper bound of $\lambda_k(\Omega)$ by means of a suitable isoperimetric deficit. To get such estimate we have to study separately the case $k = n$ and the case $1 \le k \le n - 1$. We start by recalling some properties of the eigenfunctions of $S_k$ in a ball.

Let $B_R$ be a ball of $\mathbb{R}^n$ centered at the origin with radius $R$, and $v \in C^2(B_R) \cap C(\bar{B}_R)$ be an eigenfunction of the $k$-Hessian operator in $B_R$. This means that $v$ verifies

$$(4.1) \qquad \begin{cases} S_k(D^2 v) = \lambda_k(B_R)(-v)^k & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

It is known that $v$ is a negative, convex, radially increasing smooth function. We have that:

$$(4.2) \qquad \begin{cases} v(x) = \varphi(r), & r = |x|, \ x \in B_R, \\ \varphi < 0 \text{ in } [0, R[, & \varphi(R) = 0, \\ \varphi' > 0 \text{ in } ]0, R], & \varphi'(0) = 0 \end{cases}$$

(see [3], [6]).

### 4.1. Stability estimates: the case of Monge-Ampère operator.
Let us consider problem (4.1) with $k = n$ and $B_R = \Omega^*_{n-1}$, where $\Omega^*_{n-1}$ is the ball centered at the origin and radius $R$ such that $W_{n-1}(\Omega) = W_{n-1}(\Omega^*_{n-1})$. Here, $v$ denotes an eigenfunction relative to $\lambda_n(\Omega^*_{n-1})$. Recall that the Faber-Krahn inequality (2.13) holds.

Together with (2.13), the following result gives a quantitative estimate of $\lambda_n(\Omega)$.

**Theorem 4.1.** *Let $\Omega$ be as in (2.1). Then*

$$(4.3) \qquad \frac{\lambda_n(\Omega) - \lambda_n(\Omega^*_{n-1})}{\lambda_n(\Omega)} \le C_\Omega \big[ |\Omega^*_{n-1}| - |\Omega| \big],$$

*where $C_\Omega = \left( \frac{\|v\|_\infty}{\|v\|_{n+1}} \right)^{n+1}$.*

*Proof.* Without loss of generality, we can suppose that the quantity in the right-hand side of (4.3) is smaller than 1. Otherwise, (4.3) is trivial. Let $R > 0$ such that $B_R = \Omega^*_{n-1}$, and define

$$u_\varepsilon(x) = \varphi(R - d_\varepsilon(x)), \quad x \in \Omega,$$

where $\varphi$ is given in (4.2) and $d_\varepsilon$ is the approximation of the distance function to the boundary of $\Omega$ given in Proposition 3.1. For any $\varepsilon > 0$, the function $u_\varepsilon$ is well defined, being $d_\varepsilon \le d \le r_\Omega$, and $r_\Omega \le R$. Last inequality is true since, by the definition of $W_{n-1}$ and using the Aleksandrov-Fenchel inequality for $j = n - 1$ and $i = 0$, we have

$$\omega_n R = W_{n-1}(\Omega^*_{n-1}) = W_{n-1}(\Omega) \ge \omega_n^{1-\frac{1}{n}} |\Omega|^{\frac{1}{n}} \ge \omega_n r_\Omega.$$

As matter of fact, denoting the function $g(t) = |Dv|_{\{v=-t\}}$, $0 \le t \le \|v\|_\infty$, by construction, the function $u_\varepsilon$ has the following properties:

(4.4)
$$
\begin{cases}
u_\varepsilon(x) \in \Phi_n^2(\Omega), \\
|Du_\varepsilon|_{\{u_\varepsilon=-t\}} \le g(t), \\
\|u_\varepsilon\|_\infty \le \|v\|_\infty, \\
u_\varepsilon(x) \to u(x) = \varphi(R - d(x)) \text{ uniformly in } \bar{\Omega}.
\end{cases}
$$

Let us define
$$
E_t = \{x \in \Omega : u < -t\}, \quad B_t = \{x \in \Omega_{n-1}^* : v < -t\}.
$$
$E_t$ is a convex set, while $B_t$ is a ball centered at the origin.

Lemma 3.2 implies that
$$
-\frac{d}{dt} W_{n-1}(E_t) \ge \frac{\omega_n}{g(t)} = -\frac{d}{dt} W_{n-1}(B_t).
$$

Together with the initial condition $W_{n-1}(E_0) = W_{n-1}(B_0)$, we have that
$$
W_{n-1}(E_t) \le W_{n-1}(B_t), \quad 0 < t < \sup(-v).
$$

Applying the Aleksandrov - Fenchel inequalities, the above inequality gives that

(4.5)
$$
P(E_t) \le P(B_t).
$$

Now, denote with $\mu(t) = |E_t|$ and $\nu(t) = |B_t|$. Using the coarea formula and the inequality (4.5), we have that
$$
-\mu'(t) = \int_{\{u=-t\}} \frac{1}{|Du|} d\mathcal{H}^{n-1} = \frac{P(E_t)}{g(t)} \le \frac{P(B_t)}{g(t)} = \int_{\{v=-t\}} \frac{1}{|Dv|} d\mathcal{H}^{n-1} = -\nu'(t),
$$

and then $\nu - \mu$ is a decreasing function. Hence,

(4.6)
$$
\int_\Omega (-u)^{n+1} dx = \int_0^{\|u\|_\infty} (n+1) t^n \mu(t) dt =
$$
$$
= \int_0^{\|v\|_\infty} (n+1) t^n \nu(t) dt - \int_0^{\|v\|_\infty} (n+1) t^n [\nu(t) - \mu(t)] dt \ge
$$
$$
\ge \int_{\Omega_{n-1}^*} v^p dx - (|\Omega_{n-1}^*| - |\Omega|) \|v\|_\infty^{n+1}.
$$

Then, from the uniform convergence of $u_\varepsilon$ to $u$ and (4.6) we get that

(4.7)
$$
\lim_{\varepsilon \to 0^+} \int_\Omega (-u_\varepsilon)^{n+1} dx = \int_\Omega (-u)^{n+1} dx \ge \int_{\Omega_{n-1}^*} (-v)^{n+1} dx - (|\Omega_{n-1}^*| - |\Omega|) \|v\|_\infty^{n+1}.
$$

On the other hand, (2.12) and (4.4) imply that
$$
\int_\Omega (-u_\varepsilon) \det(D^2 u_\varepsilon) dx = \int_0^{\|u_\varepsilon\|_\infty} dt \int_{\{u_\varepsilon=-t\}} H_{n-1}(\{u_\varepsilon = -t\}) |Du_\varepsilon|^n d\mathcal{H}^{n-1} \le
$$
$$
\le \int_0^{\|u_\varepsilon\|_\infty} g(t) dt \int_{\{u_\varepsilon=-t\}} H_{n-1}(\{u_\varepsilon = -t\}) d\mathcal{H}^{n-1} =
$$
$$
= n\omega_n \int_0^{\|u_\varepsilon\|_\infty} g(t) dt \le n\omega_n \int_0^{\|v\|_\infty} g(t) dt =
$$
$$
= \int_0^{\|v\|_\infty} dt \int_{\{v=-t\}} H_{n-1}(\{v = -t\}) |Dv|^n d\mathcal{H}^{n-1} = \int_{\Omega_{n-1}^*} (-v) \det(D^2 v) dx.
$$

Finally, putting together (4.7) and the above inequality, we have that

$$(4.8) \quad \lambda_n(\Omega) \le \liminf_{\varepsilon \to 0^*} \frac{\int_\Omega (-u_\varepsilon) \det(D^2 u_\varepsilon) dx}{\int_\Omega (-u_\varepsilon)^{n+1} dx} \le \frac{\int_{\Omega^*_{n-1}} (-v) \det(D^2 v) dx}{\int_{\Omega^*_{n-1}} (-v)^{n+1} dx - \left(|\Omega^*_{n-1}| - |\Omega|\right) \|v\|_\infty^{n+1}} =$$

$$= \frac{\lambda_n(\Omega^*_{n-1})}{1 - (|\Omega^*_{n-1}| - |\Omega|) \left(\frac{\|v\|_\infty}{\|v\|_{n+1}}\right)^{n+1}},$$

and then

$$\frac{\lambda_n(\Omega) - \lambda_n(\Omega^*_{n-1})}{\lambda_n(\Omega)} \le (|\Omega^*_{n-1}| - |\Omega|) \left(\frac{\|v\|_\infty}{\|v\|_{n+1}}\right)^{n+1}.$$

$\square$

**Remark 4.1.** If $n = 2$, the estimate (4.3) becomes

$$\frac{\lambda_2(\Omega) - \lambda_2(\Omega^\star)}{\lambda_2(\Omega)} \le \frac{C_\Omega}{4\pi} \left(P^2(\Omega) - 4\pi|\Omega|\right),$$

where $\Omega^\star = \Omega^*_1$ is the ball with the same perimeter than $\Omega$.

4.2. **Stability estimates: the case of $k$-Hessian operator, $k < n$.** Now we consider problem (4.1) with $1 \le k \le n - 1$ and $B_R = \Omega^*_k$, where $\Omega^*_k$ is the ball centered at the origin and radius $R$ such that $W_k(\Omega) = W_k(\Omega^*_k)$. As before, $v$ denotes an eigenfunction relative to $\lambda_k(\Omega^*_k)$.

**Theorem 4.2.** *Let $\Omega$ be as in (2.1), and $1 \le k \le n - 1$. Then*

$$(4.9) \qquad\qquad \frac{\lambda_k(\Omega) - \lambda_k(\Omega^*_k)}{\lambda_k(\Omega)} \le C_\Omega \left(|\Omega^*_k| - |\Omega|\right),$$

*where $C_\Omega = \left(\frac{\|v\|_\infty}{\|v\|_{k+1}}\right)^{k+1}$.*

*Proof.* We follow the lines of the proof of Theorem 4.1. First, suppose that the quantity in the right-hand side of (4.9) is smaller than 1. Let $R > 0$ be such that $B_R = \Omega^*_k$, and

$$u_\varepsilon(x) = \varphi(R - d_\varepsilon(x)), \quad x \in \Omega.$$

The function $u_\varepsilon$ is well defined, since by Aleksandrov-Fenchel inequalities we have

$$\omega_n R^{n-k} = W_k(\Omega^*_k) = W_k(\Omega) \ge \omega_n^{1 - \frac{n-k}{n}} |\Omega|^{\frac{n-k}{n}} \ge \omega_n r_\Omega^{n-k}.$$

By construction, $u_\varepsilon$ has the following properties:

$$\begin{cases} u_\varepsilon(x) \in \Phi_k^2(\Omega), \\ |Du_\varepsilon|_{\{u_\varepsilon = -t\}} \le g(t) := |Dv|_{v=-t}, \\ \|u_\varepsilon\|_\infty \le \|v\|_\infty, \\ u_\varepsilon(x) \to u(x) = \varphi(R - d(x)) \text{ uniformly in } \bar{\Omega}. \end{cases}$$

For $t \ge 0$, we set

$$E_t = \{x \in \Omega : -u > t\}, \quad B_t = \{x \in \Omega^*_k : -v > t\}.$$

$E_t$ is a convex set, while $B_t$ is a ball centered at the origin.

By Lemma 3.2 and the Aleksandrov-Fenchel inequalities, if $1 \le k < n - 1$, we have

$$-\frac{d}{dt}W_k(E_t) \ge (n-k)\frac{W_{k+1}(E_t)}{g(t)}$$

$$\ge (n-k)\omega_n^{\frac{1}{n-k-1}}\frac{W_k(E_t)^{\frac{n-k-1}{n-k}}}{g(t)},$$

and

$$-\frac{d}{dt}W_k(B_t) = (n-k)\frac{W_{k+1}(B_t)}{g(t)}$$

$$= (n-k+1)\omega_n^{\frac{1}{n-k}}\frac{W_k(B_t)^{\frac{n-k}{n-k}}}{g(t)}.$$

If $k = n - 1$, being $W_n(K) = \omega_n$, we write simply that

$$-\frac{d}{dt}W_{n-1}(E_t) \ge \frac{\omega_n}{g(t)},$$

and

$$-\frac{d}{dt}W_{n-1}(B_t) = \frac{\omega_n}{g(t)}.$$

Being $W_k(E_0) = W_k(B_0)$, by the classical comparison theorems for differential inequalities, we get that

(4.10)
$$W_k(E_t) \le W_k(B_t), \quad 0 < t < \|v\|_\infty.$$

The inequality (4.10) implies that

$$P(E_t) \le P(B_t).$$

Indeed, this is trivial if $k = 1$. In the case $2 \le k \le n - 1$, this follows using the Aleksandrov-Fenchel inequalities (2.9) in (4.10), and recalling that (2.9) holds as an equality for the sets $B_t$.

Now, reasoning similarly as in the proof of Theorem (4.1), it follows that

$$\lim_{\varepsilon \to 0^+} \int_\Omega (-u_\varepsilon)^{k+1}dx = \int_\Omega (-u)^{k+1}dx \ge \int_{\Omega_k^*} (-v)^{k+1}dx - (|\Omega_k^*| - |\Omega|)\|v\|_\infty^k.$$

Moreover, recalling the properties of $u_\varepsilon$ and observing that the level set $E_k^\varepsilon = \{u_\varepsilon < -t\}$ are contained in $E_t = \{u < -t\}$, by (2.12) we get that

(4.11)
$$\int_\Omega (-u_\varepsilon)\,S_k(D^2 u_\varepsilon)dx = \int_0^{\|u_\varepsilon\|_\infty} dt \int_{\{u_\varepsilon=-t\}} H_{k-1}(\{u_\varepsilon = -t\})|Du_\varepsilon|^k d\mathcal{H}^{n-1} \le$$

$$\le \int_0^{\|v\|_\infty} g(t)dt \int_{\{u_\varepsilon=-t\}} H_{k-1}(\{u_\varepsilon = -t\})d\mathcal{H}^{n-1} =$$

$$= n\int_0^{\|v\|_\infty} g(t)W_k(E_t^\varepsilon)dt \le n\int_0^{\|v\|_\infty} g(t)W_k(E_t)dt \le n\int_0^{\|u\|_\infty} g(t)W_k(B_t)dt =$$

$$= \int_0^{\|v\|_\infty} dt \int_{\{v=-t\}} H_{k-1}(\{v = -t\})|Dv|^k d\mathcal{H}^{n-1} = \int_{\Omega_k^*} (-v)\,S_k(D^2 v)dx.$$

Finally,

$$\lambda_k(\Omega) \leq \liminf_{\varepsilon \to 0^+} \frac{\int_\Omega (-u_\varepsilon)\, S_k(D^2 u_\varepsilon) dx}{\int_\Omega (-u_\varepsilon)^{k+1} dx} \leq \frac{\int_{\Omega_k^*} (-v)\, S_k(D^2 v) dx}{\int_{\Omega_k^*} (-v)^{k+1} dx - \left(|\Omega_k^*| - |\Omega|\right) \|v\|_\infty^k} =$$

$$= \frac{\lambda_k(\Omega_k^*)}{1 - (|\Omega_k^*| - |\Omega|)\left(\frac{\|v\|_\infty}{\|v\|_{k+1}}\right)^{k+1}},$$

and we can conclude that

$$\frac{\lambda_k(\Omega) - \lambda_k(\Omega_k^*)}{\lambda_k(\Omega)} \leq (|\Omega_k^*| - |\Omega|)\left(\frac{\|v\|_\infty}{\|v\|_{k+1}}\right)^{k+1}.$$

$\square$

**Remark 4.2.** We observe that if we choose $\Omega$ in the class of sets such that the Faber-Krahn inequality

$$\lambda_k(\Omega) \geq \lambda_k(\Omega_{k-1}^*)$$

holds (see Section 1.2), then (4.2) gives a quantitative estimate for $\lambda_k$ in terms of an isoperimetric deficit. Indeed, in such a case, being $\lambda_k(\cdot)$ decreasing with respect to the inclusion of sets, we have

$$0 \leq \frac{\lambda_k(\Omega) - \lambda_k(\Omega_{k-1}^*)}{\lambda(\Omega)} \leq \frac{\lambda_k(\Omega) - \lambda_k(\Omega_k^*)}{\lambda_k(\Omega)} \leq C_\Omega \left(|\Omega_k^*| - |\Omega|\right).$$

In the last subsection we give an estimate of $\lambda_k(\Omega)$ that generalizes the one obtained by Makai in [13] for the first eigenvalue of the Laplacian.

### 4.3. **An upper bound for the eigenvalue of** $S_k$, $1 \leq k \leq n$.

**Theorem 4.3.** *Let $\Omega$ verifies (2.1), and let $\lambda_k(\Omega)$ be the eigenvalue of the $k$-Hessian operator $S_k$ in $\Omega$, with $1 \leq k \leq n$. Then the following upper bound for $\lambda_k(\Omega)$ holds:*

$$(4.12) \qquad \lambda_k(\Omega) \leq \frac{n(k+2)}{n-k+1} \frac{P(\Omega)^{k+1}}{|\Omega|^{k+2}} W_{k-1}(\Omega).$$

*Proof.* Let $d_\varepsilon$ the sequence given in Proposition 3.1. Recall that $r_\Omega$ is the inradius of $\Omega$. By (2.12) and Lemma 3.1 we have that

$$(4.13) \quad \int_\Omega (d_\varepsilon)\, S_k(D^2 d_\varepsilon) dx = \int_0^{\|d_\varepsilon\|_\infty} dt \int_{\{d_\varepsilon = t\}} H_{k-1}(\{d_\varepsilon = t\})|Du_\varepsilon|^k d\mathcal{H}^{n-1} \leq$$

$$\leq \int_0^{r_\Omega} dt \int_{\{d_\varepsilon = t\}} H_{k-1}(\{d_\varepsilon = t\}) d\mathcal{H}^{n-1} = n \int_0^{r_\Omega} W_k(\{d_\varepsilon > t\}) dt \leq$$

$$\leq n \int_0^{r_\Omega} W_k(\Omega_t) dt \leq \frac{n}{n-k+1} \int_0^{r_\Omega} -\frac{d}{dt} W_{k-1}(\Omega_t) dt = \frac{n}{n-k+1} W_{k-1}(\Omega),$$

while, using the coarea formula and Lemma 3.3 with $f(t) = t^{k+1}$, we have that

$$(4.14) \qquad \int_\Omega d^{k+1} dx = \int_0^{r_\Omega} t^{k+1} P(\Omega_t) dt \geq P(\Omega) \int_0^{|\Omega|/P(\Omega)} t^{k+1} dt = \frac{1}{(k+2)} \frac{|\Omega|^{k+2}}{P(\Omega)^{k+1}}.$$

Hence, recalling also that $d_\varepsilon \to d$ uniformly in $\bar\Omega$, by (2.11), (4.13) and (4.14) we get

$$\lambda_k(\Omega) \leq \liminf_{\varepsilon \to 0^+} \frac{\int_\Omega d_\varepsilon\, S_k(D^2 d_\varepsilon) dx}{\int_\Omega d_\varepsilon^{k+1} dx} \leq \frac{n(k+2)}{n-k+1} \frac{P(\Omega)^{k+1}}{|\Omega|^{k+2}} W_{k-1}(\Omega).$$

$\square$

**Remark 4.3.** We emphasize two particular cases of (4.12). First, for $k = 1$ it becomes

$$\lambda_1(\Omega) \le 3 \frac{P^2(\Omega)}{|\Omega|^2},$$

that is exactly the Makai estimate contained in [13]. Moreover, for $k = n = 2$, the estimate (4.12) is

$$\lambda_2(\Omega) \le 4 \frac{P(\Omega)^4}{|\Omega|^4}.$$

**Remark 4.4.** We recall that an upper bound of the Dirichlet eigenvalue of the Monge-Ampère operator on convex smooth domain with fixed measure has been given in [1]. More precisely, the authors prove that

(4.15)                                        $\lambda_n(\Omega) \le \lambda_n(\Omega^\#).$

Furthermore, according to the invariance under volume preserving affine transformations of $\det D^2 u$, they prove that the equality holds if and only if $\Omega$ is an ellipsoid. Clearly, if $\Omega$ is a smooth convex set with fixed measure, the quantities in (4.15) remain bounded, while the right-hand side of (4.12) diverges if, for example, $P(\Omega) \to +\infty$. As matter of fact, (4.15) cannot hold for $\lambda_k(\Omega)$, since it may diverge as $P(\Omega) \to +\infty$ and $|\Omega|$ fixed, as shown by the following example.
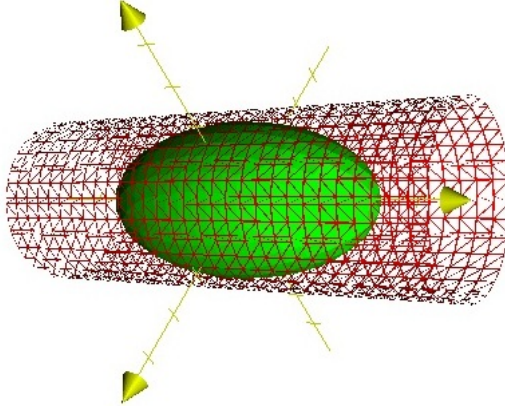


FIGURE 1. The domains $\mathcal{E}_a$ and $\mathcal{C}_{2a}$ of Example 4.1

**Example 4.1.** For sake of simplicity, we consider the case $n = 3$ and $k = 2$.

Let $\mathcal{E}_a$ be the ellipsoid $\mathcal{E}_a = \{(x, y, z) \in \mathbb{R}^3 \colon \frac{1}{a^2}(x^2 + y^2) + a^4 z^2 = 1\}$. Clearly, $|\mathcal{E}_a| = \frac{4}{3}\pi$, and $\mathcal{E}_a \cap \{z = 0\} = D_a$ is the disk of $\mathbb{R}^2$ centered at the origin with radius $a > 0$. Let $u$ be an eigenfunction of S$_2$ in $\mathcal{E}_a \subset \mathbb{R}^3$, relative to $\lambda$, and $w$ be an eigenfunction of the Monge-Ampère operator in $D_{2a} \subset \mathbb{R}^2$ relative to $\mu$, that is

$$\begin{cases} S_2(D^2 u) = \lambda(-u)^2 & \text{in } \mathcal{E}_a, \\ u = 0 & \text{on } \partial\mathcal{E}_a, \end{cases} \qquad \begin{cases} \det(D^2 w) = \mu(-w)^2 & \text{in } D_{2a}, \\ w = 0 & \text{on } \partial D_{2a}. \end{cases}$$

By the definition of S$_2$, the function

$$v(x, y, z) := w(x, y), \quad (x, y, z) \in \mathcal{C}_{2a} = \{(x, y, z) \colon (x, y) \in D_{2a}, z \in \mathbb{R}\}$$

verifies

$$\begin{cases} S_2(D^2 v) = \mu(-v)^2 & \text{in } \mathcal{E}_a, \\ v < 0 & \text{on } \partial \mathcal{E}_a. \end{cases}$$

Then, an argument based on the maximum principle for fully nonlinear elliptic equations (see [8, Theorem 17.1]) gives that

(4.16)                                   $\lambda \geq \mu.$

Finally, being

$$\mu = \frac{\int_{D_{2a}} (-w) \det D^2 w \, dx}{\int_{D_{2a}} (-w)^3 dx} \sim \frac{1}{|D_{2a}|^2},$$

by (4.16) we have that $\lambda \to +\infty$ as $a \to 0$.

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